Applying Partial Differential Equations to Information Diffusion

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1 Abstract

As social media networks have become more and more popular, it becomes worthwhile to understand how information flows through these networks. This paper examines how partial differential equations can model information diffusion by analyzing the logistic and linear models found in the paper "Modeling Information Diffusion in Online Social Networks with Partial Differential Equations" by Haiyan Wang, Feng Wang, and Kuai Xu[1].

2 Introduction

We live in a time when the way that people receive information is rapidly changing. Just decades ago, there were much stricter gatekeepers to what news circulated. People turned to their local newspaper, radio station, or television network to learn the news and didn't have much choice about which channel to watch or what newspaper to read. The rise of the internet, especially online social networks, has completely changed the way news is spread. Now with sites like Twitter and Facebook, people can see in real time what others are saying about every topic imaginable all over the world.

While these platforms provide real opportunities for small news sites or any individual to share news that would otherwise have gone unheard, they also make it much easier for false information to spread. As social media has taken off, so has research about how information spreads on these networks. A better understanding of how information spreads online could help prioritize the spread of positive information, while minimizing the impact of harmful and false content.

There has been a surge of research on diffusion of information on social networks topic in recent years. A variety of techniques have been attempted, including data mining tactics and statistical methods. While most of these models use some implementation of ordinary differential equations, very few have expanded to partial differential equations (PDEs). Using a PDE model has the benefit of considering both temporal and spatial changes in information flow, so this paper will attempt to tackle information diffusion using PDEs.

We will provide a detailed explanation of two PDE models from the paper titled "Modeling Information Diffusion in Online Social Networks with Partial Differential Equations" by Haiyan Wang, Feng Wang, and Kuai Xu. First, we will derive their models using the Conservation Law of Information Flow. The first model considers non-linear diffusion with a logistic model, and we will provide further intuition behind this model. The second model is a linear diffusion equation. Wang et al. solved all equations numerically, but we will extend their work by solving a simplified version of the second model using Sturm-Liouville theory.

3 Background

Research on information diffusion will be relevant to a wide range of social media networks, but different sites have different models of diffusion. Our research will focus on sites that have both friendship links and the ability to share/lookup information. Facebook and Twitter are perfect examples of this, as whenever someone retweets or shares something, it goes out to all of their followers. Other places, like Instagram and Reddit, friending and sharing are less prominent actions, though the capability does exist, so adjustments to the models and their parameters would be necessary. Snapchat, on the other hand, doesn't give users the ability to share other users' content, so this model would not apply.

Using PDEs to model information diffusion is much more powerful than only considering one dimension, but it might not be immediately clear what spatial variable to use. The temporal variable can simply be time since a post was published, but how does one quantify distance in a social network? An obvious way to do this is through friendship hops, or the shortest path through friends to get from one user to another. People who follow each other would be a distance of 1, whereas a friend of a friend would be distance 2, and so on. Another way to consider distance is the smallest number of shares a post would need to have by followers to get from the original poster to another user. By these metric, x is a discrete variable representing the number of friendship hops from the source. Since social media sites such as Facebook and Twitter have large networks with billions of users, we will approximate x as a continuous variable.

On sites like Twitter, Facebook, and Digg (a news aggregation site used in Wang et al.'s research), information travels in two ways. First, there is structure-based movement which uses the concept of friendship hops described earlier. This type of diffusion only occurs from viewing the posts of users one follows. If someone you follow posts something, you will see it in your feed, and then maybe you'll share it to your followers. This is structured diffusion because by posting the content, the person you follow was providing the content to only a small subset of people (i.e. their own followers). The other way that information can travel is not based on who you follow, but what information users somewhat randomly find. Certain posts land on the homepage, start to trend, or can be searched by keyword. This type of information is much less ordered and can be modeled with a random walk. Combining these two types of diffusion models can attempt to account for both types of information flow in one PDE.

4 Conservation Law of Information Flow

Conservation laws are frequently used to derive differential equation models, and Wang et al. use it as the basis for all of the models they propose. In this section, we will provide a in-depth explanation of Wang et al.'s derivation of the Conservation Law of Information Flow. Simply stated, the Conservation Law of Information Flow says that the rate of change in the magnitude of information spread is equal to the rate at which information is created and destroyed plus the influx of new information into a social space. This needs some explaining, so we will give both a mathematical and conceptual understanding of this concept. First, though, it's important to grasp the terms used in the derivation.

To simplify the model of information diffusion, information flow will be considered in a one-dimensional tube of length L. For pieces of information (i.e. a news article, social media post etc), the tube will contain interactions with the information at certain distances from the creator. The left boundary will represent the creator of the information and moving along the x-axis represents the friendship-hop distance for other accounts (friendship hops were explained in the previous section). When an account interacts with a post, this will impact the density of *influenced users* at the friendship hop distance between their account and the creator's. Wang et al. use the term *influenced user* to refer to people who have viewed a news story on their social media feed.

This measure of density of influenced users is represented by the equation $I = I(x,t)$. I has the units of density per unit length. Declaring the cross sectional area of the tube to be A we can say the information in a small section of the tube is $I(x,t)Adx$.

There are two ways that information can propagate and flow through a system. It can move in a structured way, by tracing its way through friends and followers via shares and likes. This can be thought of as information tracing its way through a graph along edges where edges are friendships and nodes are accounts becoming influenced. In this model, this type of flow, representing the rate of information creation at position x at time t will be labeled $f = f(I,x,t)$. This local growth would only be negative upon deletion of posts and is dubbed "structure-based" diffusion.

The other way information can be viewed is through more random methods, including search and trending content. This type of viewing is based almost entirely by the popularity of the content. In this model $J = J(x,t)$ represents the quantity of information crossing x at t with positive values moving right and negative values moving left. The randomness of J captures the random-walk diffusion process of "content-based" diffusion which we will be analyzed more below.

Using these terms and considering a subsection of the tube between a and b we arrive at the following equality:

$$\frac{d}{dt}\int_a^b I(x,t)Adx = AJ(a,t) - AJ(b,t) + \int_a^b f(I,x,t)Adx$$

This equality says that the rate of change in total influenced users in the subsection $a \leq x \leq b$ is the amount of users entering at a minus the users leaving at b plus the amount of information created in the subsection.

Using the fundamental theorem of calculus we can simplify $J(a,t) - J(b,t)$ to equal $-\int_a^b \frac{\partial J}{\partial x}dx$. Plugging that in to the equality above we get:

$$A\int_a^b \frac{\partial I(x,t)}{\partial t}dx = -A\int_a^b \frac{\partial J}{\partial x}dx + A\int_a^b f(I,x,t)dx$$

We can divide by A (because the tube has non-zero volume) and move everything to the left hand side to get:

$$\int_a^b \left(\frac{\partial I}{\partial t} + \frac{\partial J}{\partial x}\right)dx = \int_a^b f(I,x,t)dx$$

Because this equation is satisfied for all a and b, we know that in every region of the graph both sides of these equations are equal. Therefore, the Conservation Law of Information Flow is:

$$\frac{\partial I}{\partial t} + \frac{\partial J}{\partial x} = f(I,x,t)$$

We mentioned earlier that J resulted from content-based diffusion of information and roughly follows a random walk. Content-based diffusion is also a result of the popularity of information, because, as you can imagine, more popular information spreads farther. We will represent popularity with the variable d. The simplest version of d represents the average value of popularity at any position or time, but it could also be dependent on x or t Generally, information flows from areas with a large proportion of influenced users to areas with much lower proportions, as those are the areas with more room for growth. Based on this fact, J can be represented as the negative of the change in I (density of influenced users at a location) with respect to x. Essentially if density is getting higher, J will decrease, but if density is getting lower, J will increase. This, along with the popularity indicator d, yields:

$$J = -d\frac{\partial I}{\partial x}$$

We can then plug this into our previous equation to yield a PDE describing information flow:

$$\frac{\partial I}{\partial t} - d\frac{\partial^2 I}{\partial x^2} = f(I, x, t)$$

Or equivalently:

$$\frac{\partial I}{\partial t} = d\frac{\partial^2 I}{\partial x^2} + f(I, x, t) \tag{1}$$

Now, we can more accurately break down information flow into its two parts that we previously described. The structure-based flow is represented by $f(I, x, t)$ while the content-based flow is represented by $d\frac{\partial^2 I}{\partial x^2}$. A PDE of this form is known as a Reaction-Diffusion equation.

5 Diffusion Models

The rest of the paper will explore possible values for $f(I, x, t)$. At a high level, f reflects information flowing through followers, with respect to the density of users who have seen a post (I), number of friendship hops from the original post (x), and time since the post was created (t). First, we will provide in-depth explanations of the equations presented in the models from the Wang et al. paper. After that, we will set up a simpler version of their linear model (not from their paper) and solve it using Sturm-Liouville Theory.

5.1 Nonlinear Diffusion (Logistic Model)

The first system from the Wang et al. paper is based on the logistic model. Logistic models are most commonly used for population growth and reflect the fact that the rate of reproduction is proportional to the current population but limited by the carrying capacity.

$$\frac{\partial N}{\partial t} = rN\left(1 - \frac{N}{K}\right)$$

$N(t)$ represents the population at time t, $\partial N/\partial t$ represents the change in population over time, r represents the population growth rate, and K represents the population's carrying capacity. The meaning of carrying capacity and population growth in the social network setting will be explained in the next few paragraphs.

To understand carrying capacity for social media sites, we need to think of screen space as a limited resource that various sources compete for. There is far too much information on a social media site for a single person to digest. Users will only scroll for so long, so the carrying capacity of information diffusion is the total amount of information that can be seen by all users before they stop scrolling.

In a biological setting, the population growth rate r reflects the birth rate of new information. For social networks, r reflects the rate of information diffusion. A high r value suggests that a piece of news is being shared a lot. We want r to actually be a function of time because recent news is shared more frequently than old news. Wang et al. show that the rate at which news is shared $r(t)$ decays exponentially as time passes.

Wang et al. present the following differential equation for r:

$$\begin{aligned} \frac{dr(t)}{dt} &= -\alpha r(t) + \beta \\ r(0) &= \gamma \end{aligned} \tag{2}$$

The negative sign reflects the fact that we expect $r(t)$ to decrease by proportion α per unit time. β reflects the residual decay of information, regardless of how popular it is. Wang et al.

provide little motivation behind having separate α and β terms, but we came up with one possible interpretation. To capture the full complexity of social networks, we need to think about the different values of r for different types of information. New stories age quickly, but certain types of media, such as music videos, can remain popular for weeks or months at a time. Therefore, an accurate model needs two rate of decays:

1. The rate of decay for a specific type of information on a social network is α.
2. The natural rate of decay for all information on a social network, regardless of type, is β.

This is only one possible interpretation, but Wang et al.'s formulation of r is general enough to be sufficient for most social networking sites. Solving for r gives us:

$$\begin{aligned} \frac{dr(t)}{dt} &= -\alpha r(t) + \beta \\ \int \frac{1}{-\alpha r(t) + \beta} dr &= \int dt \\ \frac{\ln(-\alpha r(t) + \beta)}{-\alpha} &= t + C \\ -\alpha r(t) + \beta &= Ce^{-\alpha t} \\ r(t) &= Ce^{-\alpha t} + \beta/\alpha \end{aligned}$$

Solving for C using the the initial condition $r(0) = \gamma$ and plugging it in gives us:

$$r(t) = \frac{\beta}{\alpha} - e^{-\alpha t}\left(\frac{\beta}{\alpha} - \gamma\right) \tag{3}$$

Now that r has been derived, we will present the complete first model from Wang et al.'s paper (with slightly altered boundary conditions to accommodate computation in later sections):

$$\begin{aligned} \frac{\partial I}{\partial t} &= d\frac{\partial^2 I}{\partial x^2} + rI\left(1 - \frac{I}{K}\right) \\ I(x, 0) &= \phi(x), 0 \le x \le L \\ \frac{\partial I}{\partial x}(0, t) &= \frac{\partial I}{\partial x}(L, t) = 0, t \ge 0 \end{aligned} \tag{4}$$

The model is made up of three equations. The first was derived above, and the second is the initial condition when $t = 0$. The third introduces the Neumann boundary conditions meaning that the first derivative of I is set to zero on the boundaries. This makes intuitive sense as the left boundary represents the original user posting the information and the right boundary represents the maximum distance to another user from the initial user. Since all users are contained inside these boundaries, there can never be any flow of information into or out of the tube of length L. Therefore, the rate of change with respect to position (i.e. the first derivative with respect to x) will always be zero, hence the Neumann condition.

This model is too complex to solve using the techniques from class, so we will show a plot of the numerical solution from Wang et al.'s paper (Figure 1). The plot includes a comparison to real data tracking diffusion of posts from the social media site Digg. The model clearly well approximates the data.

5.2 Linear Model

While the logistic model in the previous section may get the best results, a linear model for $f(I, x, t)$ is simpler and can be solved more easily. This model is also from the Wang et al. paper. Wang

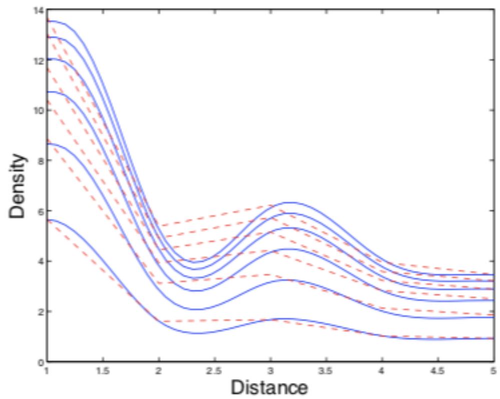

Figure 1: Logistic diffusive model prediction (blue) versus Digg data set (red) [1]

et al. noticed that the majority of influenced users (people who have seen a post) lie three or four friendship hops away from the source. The number of influenced users tapers off for accounts either closer or farther away, which makes sense as there are two competing forces affecting this number. At least initially, the number of users at each successive distance will increase exponentially. This occurs because each friend will have a number of friends, so one friend at distance one could turn into 100 friends at distance two, which could be 10,000 at distance three. Of course, some friends of friends will already be your friends and the farther the distance, the more likely a shorter path to the initial user can be found. Additionally, the farther away from the initial user, the less likely an individual user will be to have seen the content.

A concave down quadratic equation, $h(x)$, models this increase and then decrease in density of influenced users with respect to distance from the source. $h(x)$ will take the form:

$$h(x) = -(x-\rho)(x-\sigma)$$

ρ and σ are generic constants that we are not interested in. Now, with $h(x)$ representing the rate of change of influenced users with respect to position x, and $r(t)$ representing the rate of information diffusion with respect to time t (unchanged from the previous section), Wang et al. define f to be:

$$f(I,x,t) = r(t)h(x)I$$

This updated f yields the following linear diffusion equation with the same initial and boundary conditions as in the logistic equation:

$$\begin{aligned} \frac{\partial I}{\partial t} &= d\frac{\partial^2 I}{\partial x^2} - r(t)h(x)I \\ I(x,0) &= \phi(x), 0 < x < L \\ \frac{\partial I}{\partial x}(0,t) = \frac{\partial I}{\partial x}&(L,t) = 0, t > 0 \end{aligned} \tag{5}$$

Figure 2 shows that our linear model does a relatively good job fitting the Digg data.

5.3 Solving Simple Linear Model with Sturm-Liouville

The rest of this paper will go beyond the work of Wang et al. We will simplify their models to make it easier it to solve analytically. First, we can show that a simplified version of the linear

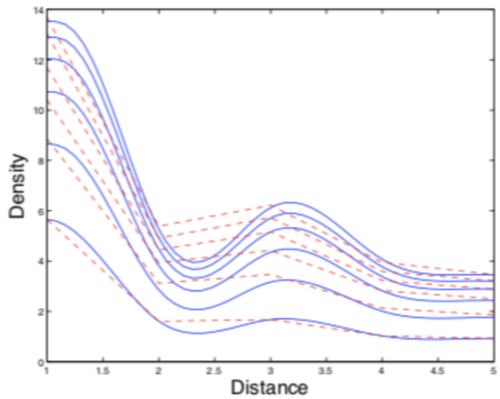

Figure 2: Linear diffusive model prediction (blue) versus Digg data set (red) [1]

diffusive model Wang et al. presented is a Sturm-Liouville problem, by making $r(t)$ a constant. The special case when $\frac{\beta}{\alpha} = \gamma$ makes r remain constant at the initial rate of influence γ, as shown below:

$$r(t) = \frac{\beta}{\alpha} - e^{-\alpha t}(\frac{\beta}{\alpha} - \gamma)$$
$$r(t) = \gamma - e^{-\alpha t}(\gamma - \gamma)$$
$$r(t) = \gamma$$

By substituting this value for $r(t)$, we now have:

$$\frac{\partial I}{\partial t} = d\frac{\partial^2 I}{\partial x^2} - \gamma h(x)I \tag{6}$$
$$\frac{\partial I}{\partial x}(0,t) = \frac{\partial I}{\partial x}(L,t) = 0, \quad t > 0 \tag{7}$$
$$I(x,0) = \phi(x), \quad 0 \leq x \leq L \tag{8}$$

5.3.1 Information Diffusion as a Sturm-Liouville Problem

Now, our goal is to show the system above is a Sturm-Liouville Problem. We can do this by substituting the ansatz $I(x,t) = e^{\lambda t}v(x)$ into equations (6)-(7).

First, we substitute the ansatz into (6), calculating the left hand side (LHS) and right hand side (RHS) separately.

$$
\begin{aligned}
LHS &= \frac{\partial I}{\partial t} \\
&= \frac{\partial}{\partial t}(e^{\lambda t}v(x)) && \text{(substituting ansatz)} \\
&= \frac{\partial}{\partial t}(e^{\lambda t})v(x) && \text{(can pull out } v(x) \text{ as not a function of t)} \\
&= \lambda e^{\lambda t}v(x)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= d\frac{\partial^2 I}{\partial x^2} - \gamma h(x)I \\
&= d\frac{\partial^2}{\partial x^2}(e^{\lambda t}v(x)) - \gamma h(x)(e^{\lambda t}v(x)) && \text{(substituting ansatz)} \\
&= de^{\lambda t}\frac{\partial^2}{\partial x^2}(v(x)) - \gamma h(x)(e^{\lambda t}v(x)) && \text{(can pull out } e^{\lambda t} \text{ as not a function of x)} \\
&= de^{\lambda t}\frac{\partial^2 v}{\partial x^2}(x) - \gamma e^{\lambda t}h(x)v(x)
\end{aligned}
$$

Thus, (6) is satisfied by the ansatz $I(x,t) = e^{\lambda t}v(x)$ if and only if:

$$
\lambda e^{\lambda t}v(x) = de^{\lambda t}\frac{\partial^2 v}{\partial x^2}(x) - \gamma e^{\lambda t}h(x)v(x)
$$

By cancelling $e^{\lambda t}$ from both sides, this is equivalent to:

$$
\lambda v(x) = d\frac{\partial^2 v}{\partial x^2}(x) - \gamma h(x)v(x)
$$

Next, we substitute the ansatz into (7), calculating each boundary condition separately.

$$
\begin{aligned}
\frac{\partial I}{\partial x}(0,t) &= \frac{\partial}{\partial x}(e^{\lambda t}v(x))\Big|_{x=0} && \text{(substituting ansatz)} \\
&= e^{\lambda t}\frac{\partial}{\partial x}(v(0)) && \text{(can pull out } e^{\lambda t} \text{ as not a function of x)} \\
&= e^{\lambda t}\frac{\partial v}{\partial x}(0) \stackrel{!}{=} 0 \implies \frac{\partial v}{\partial x}(l) = 0 && \text{(as } \frac{\partial I}{\partial x}(0,t) = 0\text{)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial I}{\partial x}(L,t) &= \frac{\partial}{\partial x}(e^{\lambda t}v(x))\Big|_{x=L} && \text{(substituting ansatz)} \\
&= e^{\lambda t}\frac{\partial}{\partial x}(v(L)) && \text{(can pull out } e^{\lambda t} \text{ as not a function of x)} \\
&= e^{\lambda t}\frac{\partial v}{\partial x}(L) \stackrel{!}{=} 0 \implies \frac{\partial v}{\partial x}(L) = 0 && \text{(as } \frac{\partial I}{\partial x}(L,t) = 0\text{)}
\end{aligned}
$$

Thus, (7) is satisfied by the ansatz if and only if:

$$
\frac{\partial v}{\partial x}(0) = \frac{\partial v}{\partial x}(L) = 0, t > 0
$$

In conclusion, the ansatz $I(x,t) = e^{\lambda t}v(x)$ satisfies (6)-(7) if and only if λ and v(x) satisfy the system:

$$d\frac{\partial^2 v}{\partial x^2}(x) = \lambda v(x) + \gamma h(x)v(x) \tag{9}$$

$$\frac{\partial v}{\partial x}(0) = \frac{\partial v}{\partial x}(L) = 0 \tag{10}$$

Now, we can divide both sides of (9) by d and substitute $\lambda_* = \frac{\lambda}{d}$ to get this system in the form of a Sturm-Liouville problem.

$$\frac{\partial^2 v}{\partial x^2}(x) = \lambda_* v(x) + \frac{\gamma}{d}h(x)v(x)$$

$$\frac{\partial v}{\partial x}(0) = \frac{\partial v}{\partial x}(L) = 0$$

$\frac{\gamma}{d}h(x)$ is a continuous function as it is a constant times a quadratic polynomial and $0 < L$, so this system is a Sturm-Liouville problem for v(x) and λ_* according to the form presented in class. This result means that we can apply all the theorems surrounding Sturm-Liouville problems we discussed over the span of the course, allowing us to draw many conclusions about the system, such as that there are infinitely many eigenvalues λ_{*n} and for each eigenvalue, there is a single linearly independent eigenfunction.

5.3.2 Further Simplification to Solve Sturm-Liouville Problem

Now that we have established that the linear diffusive model can be turned into a Sturm-Liouville problem, we will simplify the model slightly further in order to actually solve for I(x,t). We previously assumed that $r(t)$ is constant. We will now also assume that $h(x)$ is constant such that $a := r(t)h(x)$, where $a \in \mathbb{R}$.

This simplification means that $f(I, x, t) = aI$, suggesting that the information flowing through a social network is only dependent on the density of influenced users (I) and not the distance of that information from the original source and time since the post was made. This is not a realistic assumption because it implies that each tweet ever posted would eventually percolate through all Twitter users at a constant rate (like a travelling wave). Regardless, this assumption will allow us to use Sturm-Liouville theory to solve the PDE.

Using this assumption, our new PDE model to solve is as follows:

$$\begin{gathered}\frac{\partial I}{\partial t} = d\frac{\partial^2 I}{\partial x^2} - aI \\ \frac{\partial I}{\partial x}(0,t) = \frac{\partial I}{\partial x}(L,t) = 0, \quad t > 0 \\ I(x,0) = \phi(x), \quad 0 \leq x \leq L\end{gathered} \tag{11}$$

We can use our work proving this is an SL from before, as this is just a more specific case of the previous PDE, and plug this new assumption into equations (9)-(10), the step before we created λ_*, as we will want to use a different lambda transformation to create our SL problem for this scenario:

$$dv_{xx} = \lambda v(x) + av(x)$$

$$v_x(0) = v_x(L) = 0$$

Now, we can use $\tilde{\lambda} = \frac{\lambda - a}{d}$ to take advantage of the new constant and get this system into an even simpler SL problem than in the previous section:

$$\begin{aligned} \tilde{\lambda} v &= v_{xx} \\ v_x(0) = v_x(L) &= 0 \end{aligned} \tag{12}$$

Now, equation (12) is in the appropriate form to solve. According to Theorem 1 about SL problems from class, all eigenvalues are real and that there is only one linearly independent eigenvector for each eigenvalue. Therefore, we will find all solutions if we consider negative, zero, and positive numbers. Positive and zero eigenvalues only yield the trivial solution (work not shown). The negative eigenvalue case is more interesting, however.

Let $v(x) = a\cos\left(\frac{k\pi}{L}x\right) + b\sin\left(\frac{k\pi}{L}x\right)$ and $\tilde{\lambda} = \frac{-k^2\pi^2}{L^2}$, where $k > 0$ is a real number. Note that

$$v_{xx}(x) = -a\frac{k^2\pi^2}{L^2}\cos\left(\frac{k\pi}{L}x\right) - b\frac{k^2\pi^2}{L^2}\sin\left(\frac{k\pi}{L}x\right)$$

We need to verify that $\tilde{\lambda}$ and v satisfy (12). First, consider $\tilde{\lambda} v = v_{xx}$:

$$\begin{aligned} \tilde{\lambda} v &= \frac{-k^2\pi^2}{L^2}\left(a\cos\left(\frac{k\pi}{L}x\right) + b\sin\left(\frac{k\pi}{L}x\right)\right) \\ &= -a\frac{k^2\pi^2}{L^2}\cos\left(\frac{k\pi}{L}x\right) - b\frac{k^2\pi^2}{L^2}\sin\left(\frac{k\pi}{L}x\right) \\ &= v_{xx} \end{aligned}$$

Thus, the first equation in (12) is satisfied. Next, we need to show that $v_x(0) = v_x(L) = 0$ is true:

$$\begin{aligned} v_x(0) &= -a\frac{k\pi}{L}\sin(0) + b\frac{k\pi}{L}\cos(0) = b\frac{k\pi}{L} = 0 && \implies b = 0 \\ v_x(L) &= -a\frac{k\pi}{L}\sin(k\pi) + 0 \cdot \frac{k\pi}{L}\cos(k\pi) = -a\frac{k\pi}{L}\sin(k\pi) = 0 && \implies k \text{ is an integer} \end{aligned}$$

Therefore, $v_k(x)$ simplifies to $a\cos\left(\frac{k\pi}{L}x\right)$ where k is a positive integer. Since the PDE (11) is linear, we can use the superposition principle to obtain the formal series solution:

$$I(x,t) = \sum_{n=1}^{\infty} a_n e^{\lambda_n t} v_n(x) = \sum_{n=1}^{\infty} a_n e^{\lambda_n t} a\cos\left(\frac{k\pi}{L}x\right) \tag{13}$$

To solve for the a_n terms, solve for the initial condition:

$$I(x,0) = \sum_{n=1}^{\infty} a_n \cdot a\cos\left(\frac{k\pi}{L}x\right) = \phi(x)$$

By theorem 1, we can solve for the a_n's:

$$\begin{aligned}
a_n &= \frac{< \phi, v_n >}{\|v_n\|^2} \\
&= \frac{\int_0^L \phi(x) \cdot a\cos\left(\frac{k\pi}{L}x\right) dx}{\int_0^L \left(a\cos\left(\frac{k\pi}{L}x\right)\right)^2 dx} \\
&= \frac{\int_0^L \phi(x) \cdot a\cos\left(\frac{k\pi}{L}x\right) dx}{a^2 \int_0^L \cos^2\left(\frac{k\pi}{L}x\right) dx} \\
&= \frac{\int_0^L \phi(x) \cdot a\cos\left(\frac{k\pi}{L}x\right) dx}{a^2 \left[\frac{\sin(2k\pi x/L)}{4k\pi/L} + \frac{x}{2}\right]_0^L} \\
&= \frac{\int_0^L \phi(x) \cdot a\cos\left(\frac{k\pi}{L}x\right) dx}{a^2 \left[\left(\frac{\sin(2k\pi)}{4k\pi/L} + \frac{L}{2}\right) - \left(\frac{\sin(0)}{4k\pi/L} + 0\right)\right]} \\
&= \frac{2\int_0^L \phi(x) \cdot a\cos\left(\frac{k\pi}{L}x\right) dx}{a^2 L}
\end{aligned}$$

In the setting of social networks, it is safe to assume that ϕ is well-behaved such that that $\phi(x) \in L^2(0, L)$ and ϕ is continuously differentiable on $[0, L]$. Therefore, by theorem 2, $\sum_{n=1}^{\infty} a_n \cdot a\cos\left(\frac{k\pi}{L}x\right)$ uniformly converges to ϕ. Thus, the final solution to (11) is:

$$\begin{aligned}
I(x,t) &= \sum_{n=1}^{\infty} a_n e^{\lambda_n t} a\cos\left(\frac{k\pi}{L}x\right) \\
a_n &= \frac{2\int_0^L \phi(x) \cdot a\cos\left(\frac{k\pi}{L}x\right) dx}{a^2 L}
\end{aligned} \tag{14}$$

6 Conclusion

In this paper, we have analyzed both linear and logistic partial differential equation models for information diffusion in social media networks. We solved the linear system using Sturm-Liouville theory. These models can be used to predict how information will spread on social media, which is particularly important given the abundance of false information spreading on these platforms. This type of analysis can inform policies to precisely track and remove fake news. Further analysis on this topic could explore models that address competing information sources, as well information diffusion on different social media sites [1].

References

[1] H. Wang, F. Wang, K. Xu, *Modeling Information Diffusion in Online Social Networks with Partial Differential Equations*, 2013, pp. 1–15, arXiv:1310.0505